PROBLEM **1** **My World Is Turned Upside Down**

Recall from previous math courses that the reciprocal of any number x is $\frac{1}{x}$.

For example, the reciprocal of 5 is $\frac{1}{5}$ and the reciprocal of 0.5 is $\frac{1}{0.5}$, or 2.

Throughout this course you have studied many connections between polynomial functions and real numbers. Does it follow then that polynomial functions also have reciprocals? Is the reciprocal also a polynomial? Is it a function? How would the graph and table of values of $\frac{1}{f(x)}$ compare to the original function $f(x)$?

To begin answering these questions, consider the reciprocal of the basic linear function $f(x) = x$. The reciprocal can be defined as $g(x) = \frac{1}{f(x)}$, or simply $g(x) = \frac{1}{x}$.

1. Consider the graph and table of values for $f(x) = x$. The domain of $f(x)$ is $(-\infty, \infty)$. The points $(-1, -1)$ and $(1, 1)$ are shown and used to create three intervals for analysis.

x	1	2	3	4	5	6
$f(x) = x$	1	2	3	4	5	6
$g(x) = \frac{1}{x}$						
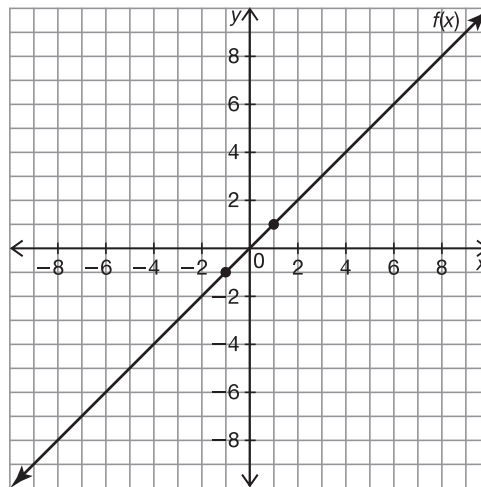
−6	−5	−4	−3	−2	−1	x
−6	−5	−4	−3	−2	−1	$f(x) = x$
						$g(x) = \frac{1}{x}$

x	−1	$-\frac{1}{2}$	$-\frac{1}{10}$	$-\frac{1}{100}$	0	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{2}$	1
$f(x) = x$	−1	$-\frac{1}{2}$	$-\frac{1}{10}$	$-\frac{1}{100}$	0	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{2}$	1
$g(x) = \frac{1}{x}$									

a. Complete the table of values for $g(x) = \frac{1}{x}$. Then plot the points and draw a smooth curve to graph $g(x)$ on the coordinate plane.

b. Describe the graph of $g(x)$. How is it similar to the graphs of other functions that you've studied? How is it different?

c. Describe the end behavior of $g(x)$. Explain your reasoning in terms of the graph, equation, and table of values.

The point at $g(0)$ is said to be *undefined* because it is impossible to divide by 0.

d. Describe $g(x)$ as x approaches 0 from the left. Explain the output behavior of the function in terms of the equation.

e. Describe $g(x)$ as x approaches 0 from the right. Explain the output behavior of the function in terms of the equation.

2. Henry and Rosie disagree about $g(x) = \frac{1}{x}$.

> Henry
> The graph and table both clearly show that it is a function.

> Rosie
> It is not a function. Every input doesn't have an output.

Who is correct? Explain your reasoning.

3. Analyze the key characteristics of $g(x) = \frac{1}{x}$.

   a. Will the graph ever intersect the horizontal line $y = 0$? Explain your reasoning in terms of the graph, table, and equation.

   b. Will the graph ever intersect the vertical line $x = 0$? Explain your reasoning in terms of the graph, table, and equation.

   c. Describe the domain and range of $g(x)$.

The function $g(x) = \frac{1}{x}$ is an example of a *rational function*. A **rational function** is any function that can be written as the ratio of two polynomials. It can be written in the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions, and $Q(x) \neq 0$. You have already seen some specific types of rational functions. Linear, quadratic, cubic, and higher order polynomial functions are types of rational functions.

Recall from your study of exponential functions that a horizontal asymptote is a horizontal line that a function gets closer and closer to, but never intersects. In this problem, the function $g(x)$ has a horizontal asymptote at $y = 0$.

The function $g(x) = \frac{1}{x}$ has a *vertical asymptote* at $x = 0$. A **vertical asymptote** is a vertical line that a function gets closer and closer to, but never intersects. The asymptote does not represent points on the graph of the function. It represents the output value that the graph approaches. An asymptote occurs for input values that result in a denominator of 0.
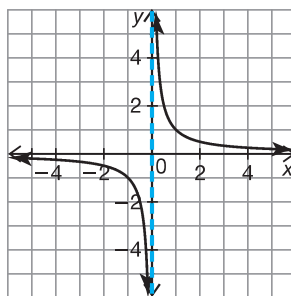
The vertical asymptote is often represented in textbooks and graphing calculators as a dashed or solid line. The convention used in this textbook is to represent asymptotes as dashed lines.

Changing the mode to "dot" on many calculators removes the asymptote from the screen. Asymptotes are often more easily viewed with a smaller viewing window. Try $[-5, 5] \times [-5, 5]$ for $g(x)$.

4. Analyze each function.

$$f(x) = x \qquad\qquad g(x) = \frac{3x}{2} \qquad\qquad h(x) = \frac{\sqrt{x}}{2x}$$

$$p(x) = \frac{3}{x} + 2 \qquad\qquad k(x) = 12 \qquad\qquad n(x) = \frac{2^x}{5}$$

$$j(x) = \frac{4x^2 + 3x + 2}{6x^3 + 10} \qquad\qquad m(x) = \frac{1}{(x + 2)(x - 3)}$$

a. Circle the rational functions.

b. Explain why the remaining functions are not rational.

c. Do you think the graphs of all rational functions will have a vertical asymptote? Explain your reasoning.

5. Use a graphing calculator to explore various rational functions of the form $p(x) = \frac{a}{x}$ where a is a constant.

a. Describe changes in the function for various a-values. Make as many conjectures as you can about the key characteristics of functions in this form.

> Consider domain, range, intercepts, end behavior, asymptotes, etc. Remember that a does not have to be an integer. Explore values between -1 and 1.

b. Abby and Natasha disagree about functions of the form $p(x) = \frac{a}{x}$ where a is a constant.

Abby

The horizontal asymptote will vary depending on the a-value.

Natasha

All rational functions of this form will have a horizontal asymptote at y = 0.

Who is correct? Explain your reasoning.

c. List several rational functions that do not have a horizontal asymptote at $y = 0$.

6. If $g(x)$ is the reciprocal function of $f(x)$, what is $f(x) \cdot g(x)$ where $g(x) \neq 0$ for any input value? Explain your reasoning.

7. The opener to this lesson provides a proof that $1 = 2$. Describe the error in the proof in terms of what you learned in this problem.